The Geometric-Algebraic Structure of the Partitioned Riemann Zeta Function

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We present a geometric-algebraic partitioning of the Riemann $\zeta(n)$ function to consider a previously unreported structural relationship between its symmetric components. A geometric interpretation of $\beta(n)$ and $\eta(n)$ is offered, exposing a coordinate ratio $K(n)$ with a remainder $k(n)$ that links discrete and known transcendental values of $\zeta(n)$. We show that $k(n)$ is well-behaved for $n \geq 2$ and approaches 1 as $n \to \infty$.

Discrete - Transcendental Coordinate Decomposition of $\zeta(n)$

$$\begin{array}{ccccc}
\text{Standard View} & & \text{scale} & & \text{Alternative View} \\
\pi^n & \cdot & (4/\pi)^n & = & 4^n \\
\div & & & & \div \\
\mathrm{R} & \cdot & (4/\pi)^n & = & \mathbf{1/h_3} \\
= & & & & = \\
\zeta(n) & & & & \zeta(n)
\end{array}$$

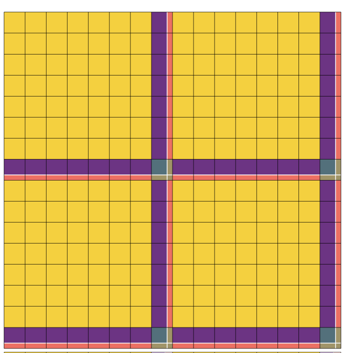
$$h_3(n) = \sum_{k=1}^{\infty} \frac{1}{(4k)^n} = \frac{1}{4^n} \sum_{k=1}^{\infty} \frac{1}{k^n} = 4^{-n} \cdot \zeta(n)$$

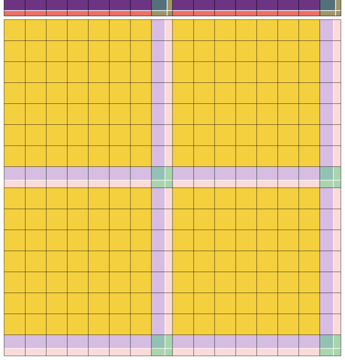
Numerical Values of the Alternative Coordinate Decomposition at n = 4

Normalized	236.95665276	15	3.04334724	1	$256 = 4^n$
	÷	÷	÷	÷	÷
scale	236.5282311	236.5282311	236.5282311	236.5282311	236.5282311
	=	=	=	=	=
Analytic	1.00181129	0.06341737	0.01286674	**0.00422782**	1.082323234

Coordinate Identifiers for the Partitioned $\zeta(n)$ Framework

Normalized	M_0	M_1	M_2	M_3	$M_0+M_1+M_2+M_3$
	÷	÷	÷	÷	÷
scale	$1/h_3$	$1/h_3$	$1/h_3$	$1/h_3$	$1/h_3$
	=	=	=	=	=
Analytic	h_0	h_1	h_2	$\mathbf{h_3}$	$\zeta(n)$





Transcendental $\zeta(4)$

$$\Phi_{\text{pink}}(4)\dagger = 2(M_2 - M_3) = 2 \cdot (3.0433472365 - 1) = 4.086694473$$

$$2(h_2 - h_3) = \zeta(4) \cdot \frac{\Phi_{\text{pink}}(4)}{4^4} = 4.086694473 \cdot h_3$$

$$\Psi_{\text{purple}}(4) = (M_0 - M_2) - (M_1 + M_3)(M_1 - M_3) = 233.913305527 - 224 = 9.913305527$$

$$\beta(4) - \eta(4) = \zeta(4) \cdot \frac{\Psi_{\text{purple}}(4)}{4^4}$$

$$\beta(4) - \eta(4) = 9.913305527 \cdot h_3$$

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Discrete $\zeta(4)$: $M_2 = \left\lfloor \left(\frac{4}{3}\right)^4 \right\rfloor + 0.5$

$$\Phi_{\text{salmon}}(4) = 2(M_2 - M_3) = 2 \cdot (3.5 - 1) = 5$$

$$2(h_2 - h_3) = \zeta(4) \cdot \frac{\Phi_{\text{salmon}}(4)}{4^4} = 5 \cdot h_3$$

$$\Psi_{\text{lavender}}(4) = (M_0 - M_2) - (M_1 + M_3)(M_1 - M_3) = 233 - 224 = 9$$

$$\beta(4) - \eta(4) = \zeta(4) \cdot \frac{\Psi_{\text{lavender}}(4)}{4^4} = 9 \cdot h_3$$

$$\beta(4) - \eta(4) = 9 \cdot h_3$$

† The subscripted variables Φ_{color} and Ψ_{color} represent the explicit geometric-algebraic partitions of the 4^n coordinate matrix. For their analytical equivalents in terms of the Dirichlet beta function $\beta(n)$, the Dirichlet eta function $\eta(n)$, and the Hurwitz Zeta function, see the **Analytical Translation Key** in Appendix A.

* This work is the result of cooperative interaction between the authors and several Gemini 1.5 Pro LLM Agents.

1 Tables for Transcendental and Discrete $\zeta(n)$ values for $n = 2 \to 8$

n	$M_0 - M_2$	$-$	$(M_1 + M_3)(M_1 - M_3)$	$=$	$\frac{1}{4} \cdot \Psi_{\text{transcendental}}(n)$
2	8.909444945	$-$	8	$=$	0.909444945
3	51.58870033	$-$	48	$=$	3.58870033
4	233.913305527	$-$	224	$=$	9.913305527
5	983.73836643	$-$	960	$=$	23.73836643
6	4020.88029404	$-$	3968	$=$	52.88029404
7	16241.0996116	$-$	16128	$=$	113.0996116
8	65260.0803677	$-$	65024	$=$	236.0803677

n	$M_0 - M_2$	$-$	$(M_1 + M_3)(M_1 - M_3)$	$=$	$\frac{1}{4} \cdot \Psi_{\text{discrete}}(n)$
2	8	$-$	8	$=$	0
3	51	$-$	48	$=$	3
4	233	$-$	224	$=$	9
5	983	$-$	960	$=$	23
6	4020	$-$	3968	$=$	52
7	16241	$-$	16128	$=$	113
8	65260	$-$	65024	$=$	236

2 The Coordinate Ratio $K(n)$ and the Coordinate Remainder $k(n)$

$K(n)$ is defined as the ratio of the even-parity differences to the odd-parity differences:

$$K(n) = \frac{M_0 - M_2}{M_1 - M_3} = \frac{h_0 - h_2}{h_1 - h_3} = 2^n \cdot \frac{\beta(n)}{\eta(n)} \tag{1}$$

For $n = 4$, the ratio is:

$$K(4) = 2^4 \cdot \frac{\beta(4)}{\eta(4)} \approx 16.70809325 \tag{2}$$

$k(n)$, the remainder of $K(n)$, is therefore:

$$k(n) = K(n) - 2^n = 2^n \cdot \left(\frac{\beta(n)}{\eta(n)} - 1 \right) \tag{3}$$

For $n = 4$, the remainder is:

$$k(4) = K(4) - 2^4 \approx 0.70809325 \tag{4}$$

3 Asymptotics and Convergence Rate of the Coordinate Remainder k(n)

We analyze the asymptotic behavior of the coordinate remainder $k(n)$ by first expanding the coordinate ratio $K(n)$, defined as:

$$K(n) = 2^n \frac{\beta(n)}{\eta(n)} \tag{5}$$

For large values of n, the Dirichlet beta function $\beta(n)$ and the Dirichlet eta function $\eta(n)$ are expanded asymptotically as:

$$\beta(n) = 1 - \frac{1}{3^n} + O\left(\frac{1}{5^n}\right) \tag{6}$$

$$\eta(n) = 1 - \frac{1}{2^n} + \frac{1}{3^n} + O\left(\frac{1}{4^n}\right) \tag{7}$$

We represent their quotient as a rational expression:

$$\frac{\beta(n)}{\eta(n)} = \left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{2^n} + \frac{1}{3^n}\right)^{-1} \tag{8}$$

Applying a first-order Taylor expansion to the inverse term yields:

$$\frac{\beta(n)}{\eta(n)} \approx \left(1 - \frac{1}{3^n}\right)\left(1 + \frac{1}{2^n} - \frac{1}{3^n}\right) = 1 + \frac{1}{2^n} - 2\left(\frac{1}{3^n}\right) + O\left(\frac{1}{4^n}\right) \tag{9}$$

The coordinate remainder $k(n)$ is defined as:

$$k(n) = 2^n \left(\frac{\beta(n)}{\eta(n)} - 1 \right) \tag{10}$$

Substituting our joint expansion into this definition gives:

$$k(n) \approx 2^n \left(1 + \frac{1}{2^n} - 2\left(\frac{1}{3^n}\right) + O\left(\frac{1}{4^n}\right) - 1 \right) \tag{11}$$

Distributing the factor of 2^n resolves the leading terms and establishes the decay rate:

$$k(n) \approx 1 - 2\left(\frac{2}{3}\right)^n + O\left(2^{-n}\right) \tag{12}$$

Taking the limit as n approaches infinity shows that the coordinate remainder converges to 1:

$$\lim_{n\to\infty} k(n) = 1 \tag{13}$$

4 Matrix-Coordinate Verification

We verify the coordinate relationship at $n = 4$ using both the transcendental and discrete representations, demonstrating how $k(n)$ scales with the coordinate difference:

$$\text{Term}_{\text{transcendental}}(4) = \frac{1}{2}\Psi_{\text{transcendental}}(4) = \frac{1}{2}k(4)_{\text{transcendental}}(2^4 - 2) = \frac{1}{2} \cdot 9.913305527 = 4.95665276 \tag{14}$$

$$\text{Term}_{\text{discrete}}(4) = \frac{1}{2}\Psi_{\text{discrete}}(4) = \frac{1}{2}k(4)_{\text{discrete}}(2^4 - 2) = \frac{1}{2} \cdot 9 = 4.5 \tag{15}$$

5 Numerical Tables and Coordinate Recovery

The terms, $M_1 = 2^n - 1$ and $M_3 = 1$ for $\zeta(n)$ are known. The terms M_0 and M_2 can be calculated from $k(n)$ as shown in the following formulas and tables.

$$\text{Term}_{\text{transcendental}}(n) = \frac{1}{2}k(n)_{\text{transcendental}}(2^n - 2), \quad \text{Term}_{\text{discrete}}(n) = \frac{1}{2}k(n)_{\text{discrete}}(2^n - 2) \tag{16}$$

The difference, diff $= \text{Term}_{\text{transcendental}} - \text{Term}_{\text{discrete}}$, governs the residues of M_0 and M_2 relative to their integer floors:

n	$K(n)$	$k(n)$	$\text{Term}_{\text{transcendental}}(n)$	$\text{Term}_{\text{discrete}}(n)$	Difference	$0.5 + \text{diff}$	$0.5 - \text{diff}$
2	4.45472247	0.45472247	0.45472247	0.5	−0.04527753	**0.45472247**	**0.54527753**
3	8.59811672	0.59811672	1.79435017	1.5	0.29435017	**0.79435017**	**0.20564983**
4	16.70809325	0.70809325	4.95665276	4.5	0.45665276	**0.95665276**	**0.04334724**
5	32.79127888	0.79127888	11.86918322	11.5	0.36918322	**0.86918322**	**0.13081678**
6	64.85290797	0.85290797	26.44014702	26.5	−0.05985298	**0.44014702**	**0.55985298**
7	128.89761597	0.89761597	56.54980582	56.5	0.04980582	**0.54980582**	**0.45019418**
8	256.92945027	0.92945027	118.04018383	118.5	−0.45981617	**0.04018383**	**0.95981617**

Comparing transcendental M_0 and M_2 values to their $k(n)$ derived remainders, then scaling the normalized coordinate to the analytic coordinate by multiplying by h_3, we conclude the geometric analysis of $\zeta(n)$.

n	M_0 transcendental	$0.5 + \text{diff}$	M_2 transcendental	$0.5 - \text{diff}$	$h_3(n)$	$h_0(n) = M_0 \cdot h_3$	$h_2(n) = M_2 \cdot h_3$
2	10.45472247	**0.45472247**	1.54527753	**0.54527753**	0.10300914	1.07693581	0.15917750
3	53.79435017	**0.79435017**	2.20564983	**0.20564983**	0.01878214	1.01037297	0.04142682
4	236.95665276	**0.95665276**	3.04334724	**0.04334724**	0.00422783	1.00181129	0.01286674
5	987.86918322	**0.86918322**	4.13081678	**0.13081678**	0.00101262	1.00034375	0.00418296
6	4026.44014702	**0.44014702**	5.55985298	**0.55985298**	0.00024838	1.00007886	0.00138097
7	16248.54980582	**0.54980582**	7.45019418	**0.45019418**	0.00006154	1.00001870	0.00045847
8	65270.04018383	**0.04018383**	9.95981617	**0.95981617**	0.00001531	1.00000454	0.00015251

6 The Floor Boundary Inequality and Conjecture

Proposition 1. *Let* $M_2(n) = \frac{h_2(n)}{h_3(n)}$ *where* $h_2(n) = \sum_{k=1}^{\infty}(4k-1)^{-n}$ *and* $h_3(n) = \sum_{k=1}^{\infty}(4k)^{-n}$. *For all* $n \geq 2$, *we establish the upper bound:*

$$M_2(n) < \left(\frac{4}{3}\right)^n \tag{17}$$

and prove:

$$(4/3)^n - M_2(n) \leq 1 \tag{18}$$

Proof. By definition, the components $h_3(n)$ and $h_2(n)$ are expressed as:

$$h_3(n) = \sum_{k=1}^{\infty}(4k)^{-n} = 4^{-n}\zeta(n) \tag{19}$$

$$h_2(n) = \sum_{k=1}^{\infty}(4k-1)^{-n} = 3^{-n} + \sum_{k=1}^{\infty}(4k+3)^{-n} \tag{20}$$

We represent the tail sum of $h_2(n)$ as a scaled Hurwitz zeta tail:

$$\sum_{k=1}^{\infty}(4k+3)^{-n} = \sum_{k=1}^{\infty}\left[4\left(k+\frac{3}{4}\right)\right]^{-n} = 4^{-n}\sum_{k=1}^{\infty}\left(k+\frac{3}{4}\right)^{-n} \tag{21}$$

Substituting these expressions back into the definition of $M_2(n)$ yields:

$$M_2(n) = \frac{3^{-n} + 4^{-n}\sum_{k=1}^{\infty}(k+3/4)^{-n}}{4^{-n}\zeta(n)} = \frac{1}{\zeta(n)}\left[\left(\frac{4}{3}\right)^n + \sum_{k=1}^{\infty}\left(k+\frac{3}{4}\right)^{-n}\right] \tag{22}$$

To analyze this boundary, we evaluate the difference:

$$\left(\frac{4}{3}\right)^n - M_2(n) = \left(\frac{4}{3}\right)^n - \frac{1}{\zeta(n)}\left[\left(\frac{4}{3}\right)^n + \sum_{k=1}^{\infty}\left(k+\frac{3}{4}\right)^{-n}\right] \tag{23}$$

$$= \frac{1}{\zeta(n)}\left[\left(\frac{4}{3}\right)^n(\zeta(n)-1) - \sum_{k=1}^{\infty}\left(k+\frac{3}{4}\right)^{-n}\right] \tag{24}$$

Using the identity $\zeta(n) - 1 = \sum_{k=2}^{\infty}k^{-n} = \sum_{k=1}^{\infty}(k+1)^{-n}$, we scale the leading term:

$$\left(\frac{4}{3}\right)^n(\zeta(n)-1) = \sum_{k=1}^{\infty}\left(\frac{4}{3(k+1)}\right)^n = \sum_{k=1}^{\infty}\left(\frac{4}{3k+3}\right)^n \tag{25}$$

Substituting this back into the difference equation gives:

$$\left(\frac{4}{3}\right)^n - M_2(n) = \frac{1}{\zeta(n)}\sum_{k=1}^{\infty}\left[\left(\frac{4}{3k+3}\right)^n - \left(\frac{4}{4k+3}\right)^n\right] \tag{26}$$

For $k \geq 1$, the terms satisfy the inequality:

$$\frac{4}{3k+3} > \frac{4}{4k+3} > 0 \tag{27}$$

Since each term in the summation is positive, the difference is positive for all $n \geq 2$:

$$M_2(n) < \left(\frac{4}{3}\right)^n \tag{28}$$

The dominant $k = 1$ term governs the exponential decay rate $\Theta((2/3)^n)$:

$$\left(\frac{4}{3}\right)^n - M_2(n) \approx \frac{1}{\zeta(n)}\left[\left(\frac{2}{3}\right)^n - \left(\frac{4}{7}\right)^n\right] \tag{29}$$

□

Numerical Boundary Verification of the Coordinate Floor for $n = 2 \to 8$

n	Rational Base $\left(\frac{4}{3}\right)^n$	Coordinate Ratio $M_2(n)$	Difference $\delta(n)$	Floor Distance $\left(\frac{4}{3}\right)^n - \left\lfloor\left(\frac{4}{3}\right)^n\right\rfloor$
2	1.77777778	1.54527753	0.23250025	0.77777778
3	2.37037037	2.20564983	0.16472054	0.37037037
4	3.16049383	3.04334724	0.11714659	0.16049383
5	4.21399177	4.13081678	0.08317499	0.21399177
6	5.61865569	5.55985298	0.05880271	0.61865569
7	7.49154092	7.45019418	0.04134674	0.49154092
8	9.98872123	9.95981617	0.02890506	0.98872123

The shared integer floor depends on the distribution of the fractional parts of powers of 3/2 (Mahler's 3/2 problem).

Conjecture 1. *For all $n \geq 2$:*

$$\lfloor M_2(n) \rfloor = \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor \tag{30}$$

The relation has been verified computationally in the range $2 \leq n \leq 500,000$.

7 Analytical Continuation and Future Paths

While the discrete coordinate predictor $M_2(n)$ is defined over the integer domain, the partitioning framework possesses a natural analytic continuation to the complex domain $s = \sigma + it$. In this broader space, the coordinate ratio becomes a meromorphic function:

$$M_2(s) = \frac{h_2(s)}{h_3(s)} = \frac{\zeta(s, 3/4)}{\zeta(s)} \tag{31}$$

This continuation exposes a geometric connection to the Riemann Hypothesis. Because $h_3(s)$ vanishes at the non-trivial zeros of the Riemann Zeta function, the complex coordinate ratio $M_2(s)$ must possess poles at these critical coordinates on the line $\sigma = 1/2$. Analytically, the interlocking zero-loci of the coordinate difference $h_0(s) - h_2(s)$ (the Dirichlet beta function) and the pole-trajectories of $M_2(s)$ suggest a structured geometric phase alignment in the critical strip. Investigating the phase alignment and trajectory dynamics of these complex coordinates off the critical line represents a path for future research.

A Analytical Translation Key

A.1 Coordinate and Component Mappings

The partitioned analytic coordinates $h_k(n)$ correspond to specific combinations of the Hurwitz zeta function $\zeta(s,a) = \sum_{m=0}^{\infty}(m+a)^{-s}$:

$$h_3(n) = \sum_{k=1}^{\infty} \frac{1}{(4k)^n} = 4^{-n}\zeta(n) \tag{32}$$

$$h_1(n) = \sum_{k=1}^{\infty} \frac{1}{(4k-2)^n} = (2^{-n} - 4^{-n})\zeta(n) \tag{33}$$

$$h_0(n) = \sum_{k=1}^{\infty} \frac{1}{(4k-3)^n} = 4^{-n}\zeta(n, 1/4) \tag{34}$$

$$h_2(n) = \sum_{k=1}^{\infty} \frac{1}{(4k-1)^n} = 4^{-n}\zeta(n, 3/4) \tag{35}$$

The symmetric analytic coordinate differences map directly to classical functions:

- **Even-Parity Difference (First Difference):** Represents the Dirichlet beta function $\beta(n) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-n}$:

$$h_0(n) - h_2(n) = \beta(n) \tag{36}$$

- **Odd-Parity Difference (Second Difference):** Represents a scaled form of the Dirichlet eta function $\eta(n) = (1 - 2^{1-n})\zeta(n)$:

$$h_1(n) - h_3(n) = 2^{-n}\eta(n) = 2^{-n}(1 - 2^{1-n})\zeta(n) \tag{37}$$

A.2 Geometric-Algebraic Variable Mappings

The subscripted variables $\Phi_{\text{transcendental}}$ and $\Psi_{\text{transcendental}}$ introduced on Page 1 correspond to explicit divisions of the 4^n coordinate matrix:

Variable	**Geometric Matrix Color**	**Analytical Definition**
$\Phi_{\text{transcendental}}(n)$	Pink (Transcendental Matrix)	$2(M_2 - M_3) = \frac{2(h_2 - h_3)}{h_3}$
$\Psi_{\text{transcendental}}(n)$	Purple (Transcendental Matrix)	$(M_0 - M_2) - (M_1^2 - M_3^2) = \frac{4^n}{\zeta(n)}(\beta(n) - \eta(n))$
$\Phi_{\text{discrete}}(n)$	Salmon (Discrete Matrix)	$2(M_2 - M_3)$ where $M_2 = \lfloor (4/3)^n \rfloor + 0.5$
$\Psi_{\text{discrete}}(n)$	Lavender (Discrete Matrix)	$(M_0 - M_2) - (M_1^2 - M_3^2)$ (discrete)

A.3 $K(n)$ and the Dirichlet Ratio

$K(n)$ is a scaled ratio of the Dirichlet beta and eta functions:

$$K(n) = \frac{M_0 - M_2}{M_1 - M_3} = \frac{h_0 - h_2}{h_1 - h_3} = \frac{\beta(n)}{2^{-n}\eta(n)} = \frac{2^n\beta(n)}{(1 - 2^{1-n})\zeta(n)} \tag{38}$$